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ON THE CAUCHY PROBLEM FOR THE HELMHOLTZ EQUATION

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Abstract: The research explores the ongoing assessment and reliability of the solution to the Cauchy problem related to the Helmholtz equation within a specific domain, using known values from a smooth segment of the domain's boundary as a reference point. This situation falls within the realm of mathematical physics where the solutions discovered do not consistently rely on the initial conditions. Highlighting practical applications, it is crucial not only to find an approximate solution but also to ascertain its derivative. Assuming that a solution exists and is continuously differentiable in a nearby region, accurate Cauchy data is scrutinized. A concrete formula has been developed to express both the solution and its derivative, along with a regularization approach for scenarios where ongoing approximations of the initial Cauchy data are provided under certain conditions, featuring a designated error threshold in the uniform metric rather than using the original data. Evaluations confirming the stability of the solution to the classical Cauchy problem have been presented.

Keywords: Integral formula, Ill-posed Cauchy problem, Carleman function, Sustainability estimates, Uniform convergence.

AMS Subject Classification: 35J46, 35J56.

1. Introduction

The study focuses on the investigation and evaluation of the stability and extension of solutions pertaining to the Cauchy problem associated with the Helmholtz equation. This analysis is performed within a certain area, utilizing data from predefined values on a smooth segment of the domain's boundary. The problem is situated within a broader context of mathematical physics, particularly where solutions exhibit a lack of continuous dependence on the initial conditions. When addressing practical concerns, the goal extends beyond merely identifying an approximate solution; it also involves its derivative. It is acknowledged that a solution to the problem exists and is continuously differentiable across a closed domain that aligns with the Cauchy data. Under these circumstances, an explicit formula facilitating the continuation of both the solution and its derivative is derived. Additionally, a regularization formula is proposed for scenarios where continuous approximations possess a specific uniform metric error, serving as an alternative to the original Cauchy data. Certainty measures related to the classical interpretation of Cauchy problem solutions have been achieved. The examined problem is classified among ill-posed problems in mathematical physics. Tikhonov A.N. [45] elucidated the inherent nature of such challenges, emphasizing their practical significance, noting that by limiting potential solution classes to a compact set, one can ascertain stability through existence and uniqueness, thus ensuring the sustainability of the task. Formulas that facilitate the discovery of solutions to the elliptic equation when only partial Cauchy data is available are termed Carleman-type formulas. In [14], Carleman formulated such a method, providing a solution to the Cauchy-Riemann equations in a specific domain shape. This concept was further advanced by G.M. Goluzin and V.I. Krylov [16], who developed a formula enabling the evaluation of analytic functions using data exclusively from boundary segments. Central to this research, as well as related studies, is the Helmholtz equation, which yields different solutions based on the operational spaces involved. In 1977, the esteemed mathematician Sh. Yarmukhamedov introduced a technique for constructing a family of fundamental solutions, resulting in explicit formulas for recovering solutions of elliptic problems from their Cauchy data on a boundary segment. Formulas with these characteristics are referred to as Carleman matrices (see, for instance [1-2], [44] and [48].

The Cauchy problem associated with most elliptic equations is characterized by having a unique solution, indicating that this problem is indeed solvable for a data set that is dense across the entirety of the domain. However, this data set lacks closure. Consequently, the theory surrounding the solvability of such problems is quite intricate. Building on these findings, explicit regularized solutions for the Cauchy problem have been determined for various factorizations of the Helmholtz operator [18–26]. Additionally, readers can explore several boundary value problems in greater detail in references [3-13], [15], [17], [27-43] and [46-47]. The complexity of the Cauchy problem for elliptic equations underpins the challenge faced when attempting to derive general results applicable across varied scenarios. It revolves around the relationship between the data set's denseness and the closure, which introduces subtlety into the existence and uniqueness of solutions. Exploring these interactions can lead to a deeper understanding of how perturbations in data affect the overall solution landscape, thereby illuminating paths towards effective regularization techniques.

Recent advances in the development of explicit solutions for the Cauchy problem have intersected with various factorization methods of the Helmholtz operator. These methodologies not only enhance the computational efficiency of solving elliptic equations but also support the stability and robustness of the solutions produced. The exploration of these factorization techniques is crucial, as they form the backbone of numerical algorithms that are widely employed in practical applications ranging from physics to engineering. Moreover, the references regarding boundary value problems provide a comprehensive overview of complementary theory and applications that further elucidate elliptic equations' behavior. Such investigations allow for a broader context within which the Cauchy problem can be analyzed, ultimately leading to a more comprehensive grasp of its solvability and the implications of non-unique solutions in various fields. Each reference serves as a stepping stone to uncover deeper insights and analytical tools that can be utilized to tackle real-world problems effectively.

The Helmholtz equation, a pivotal component in various fields such as acoustics, electromagnetism, and quantum mechanics, poses significant analytical challenges. To tackle these, the development of a fundamental solution set is essential. This entails identifying a specific entire function that encapsulates the essential characteristics of the equation's solutions. Such functions not only facilitate the understanding of the equation's behavior but also serve as a foundation for more complex mathematical and physical interpretations. In particular, the entire function must exhibit properties such as growth rates, analytic continuation, and singularity behavior, which are critical for establishing a comprehensive solution framework. The interplay between the solutions and the boundary conditions becomes increasingly vital, as it directly influences the uniqueness and stability of the derived solutions. Moreover, the utilization of complex analysis techniques, including contour integration and residue theorems, can significantly aid in deriving these fundamental solutions. Exploring the ramifications of the fundamental solution set allows for broader applications across multiple scientific disciplines. In acoustics, for example, the ability to model wave propagation through various media hinges on these mathematical constructs. As such, advancing our understanding of the Helmholtz equation through the lens of entire functions paves the way for innovative methodologies in both theoretical and applied research.

Suppose $\mathbb{R}^2$ be a real Euclidean space,

$$\xi = (\xi_1, \xi_2) \in \mathbb{R}^2,\ \eta = (\eta_1, \eta_2) \in \mathbb{R}^2,\ \alpha = |\eta_1 - \xi_1|,\ r = |\eta - \xi|.$$

Consider a set Θ that is a subset of $\mathbb{R}^2$, which denotes a real Euclidean space. Let Ω represent a bounded simply connected region within $\mathbb{R}^2$, having a boundary that is piecewise smooth. This boundary consists of the line in the plane described by $\eta_2 = 0$ and a smooth curve Σ situated in the half-plane where $\eta_2 > 0$. We can express the overall boundary of Θ as $\partial\Theta = \Sigma \cup \Omega$.

We analyze the Helmholtz equation within the specified domain Θ defined by

$$\Delta W(\eta) + \Lambda^2 W(\eta) = 0, \tag{1}$$

here Λ is the positive number, $\Delta = \sum_{j=1}^{2} \partial_{\xi_j}^2$.

Let $\Re(w)$ denote an entire function that yields real values when w is real (where $= u + iv$, with both u and v being real numbers), and adheres to the following criterion:

$$|\Re(u) \neq 0, \sup_{v \geq 1} \left| v^p \Re^{(p)}(w) \right| = K(u,p) < \infty,\ u \in (-\infty, \infty),\ p = \overline{0,2}. \tag{2}$$

Consider the function $\Psi(\eta, \Lambda; \xi)$ as established through the equation below:

$$\Psi(\eta, \Lambda; \xi) = -\frac{1}{2\pi K(\xi_2)} \int_0^\infty \operatorname{Im} \frac{\Re(w)}{w - \xi_2} \frac{u \Im_0(\Lambda u)}{\sqrt{u^2 + \alpha^2}} du, \quad \eta \neq \xi, w = i\sqrt{u^2 + \alpha^2} + \eta_2, \tag{3}$$

here $\Im_0(\Lambda u) -$ is the Bessel function.

The function $\Psi(\eta, k; \xi)$ can be expressed as

$$\Psi(\eta, \Lambda; \xi) = \vartheta(\Lambda r) + \psi(\eta, \Lambda; \xi). \tag{4}$$

where $\vartheta(\Lambda r) = -\frac{i}{4} H_0^{(1)}(\Lambda r)$ is the fundamental solution of the Helmholtz equation, $\psi(\eta, \Lambda; \xi) -$ is the regular solution of the Helmholtz equation with respect to the variable η, including the point $\eta = \xi$.

The Cauchy problem. Suppose $W(\eta) \in C^2(\Theta) \cap C^1(\Theta)$ and

$$W(\eta)|_S = f(\eta),\ \left.\frac{\partial W(\eta)}{\partial n}\right|_S = g(\eta),\ \eta \in S. \tag{5}$$

The Cauchy problem involves specific initial data and aims to determine the function $W(\eta) = W(\eta_1, \eta_2) \in C^2(\Theta) \cap C^1(\Theta)$ in Θ based on its values $f(\eta)$ и $g(\eta)$ on the boundary $\partial\Theta$.

The primary aim is to achieve an approximate resolution of the Helmholtz equation, simultaneously evaluating the stability of these solutions in relation to the Cauchy problem. As the analysis becomes more detailed, it will yield actionable outcomes, resulting in significant advantages for mathematical physics and different branches of the natural sciences.

By selecting the alternative

$$\Re(w) = e^{\lambda w},\ \Re(\xi_2) = e^{\lambda \xi_2},\ \lambda > 0, \tag{6}$$

in equation (2), we arrive at the following integral formulation

$$\Psi_\lambda(\eta, \Lambda; \xi) = -\frac{e^{-\lambda \xi_2}}{2\pi} \int_0^\infty \operatorname{Im} \frac{e^{\lambda w}}{w - \xi_2} \frac{u \Im_0(\Lambda u)}{\sqrt{u^2 + \alpha^2}} du, \tag{7}$$

For a function $W(\eta) \in C^2(\Theta) \cap C^1(\Theta)$ and any $\xi \in \Theta$, the following Green's integral formula holds:

$$W(\xi) = \int_{\partial\Theta}[\partial_n W(\eta)\Psi_\lambda(\eta, \Lambda; \xi) - W(\eta)\partial_n \Psi_\lambda(\eta, \Lambda; \xi)]ds_\eta,\ \xi \in \Theta, \tag{8}$$

Theorem 1. Let $W(\eta) \in C^2(\Theta) \cap C^1(\Theta)$ and

$$|W(\eta)| + |\partial_n W(\eta)| \leq K,\ \eta \in \Omega. \tag{9}$$

If

$$W_\lambda(\xi) = \int_\Sigma[g(\eta)\Psi_\lambda(\eta, \lambda; \xi) - f(\eta)\partial_n \Psi_\lambda(\eta, \lambda; \xi)]ds_\eta,\ \xi \in \Theta, \tag{10}$$

consequently, the subsequent evaluations are accurate

$$|W(\xi) - W_\lambda(\xi)| \leq C(\Lambda, \xi)\lambda K e^{-\lambda\xi_2},\ \lambda > 1,\ \xi \in \Theta, \tag{11}$$

$$\left|\partial_{\xi_j} W(\xi) - \partial_{\xi_j} W_\lambda(\xi)\right| \leq C(\Lambda, \xi)\lambda K e^{-\lambda\xi_2},\ \lambda > 1,\ \xi \in \Theta,\ j = \overline{1,2}. \tag{12}$$

Here, $C(\Lambda, \xi)$ and further, signifies bounded functions defined on compact subsets of Θ.

Proof. To begin, let's determine the estimate (11). Utilizing the integral expression (8) along with the equation (10), we can obtain the subsequent results:

$$\begin{aligned} W(\xi) &= \int_{\partial\Theta}[\partial_n W(\eta)\Psi_\lambda(\eta, \Lambda; \xi) - W(\eta)\partial_n \Psi_\lambda(\eta, \Lambda; \xi)]ds_\eta = \\ &= \int_\Sigma[\partial_n W(\eta)\Psi_\lambda(\eta, \Lambda; \xi) - W(\eta)\partial_n \Psi_\lambda(\eta, \Lambda; \xi)]ds_\eta + \\ &+ \int_\Omega[\partial_n W(\eta)\Psi_\lambda(\eta, \Lambda; \xi) - W(\eta)\partial_n \Psi_\lambda(\eta, \Lambda; \xi)]ds_\eta = \\ &= W_\lambda(\xi) + \int_\Omega[\partial_n W(\eta)\Psi_\lambda(\eta, \Lambda; \xi) - W(\eta)\partial_n \Psi_\lambda(\eta, \Lambda; \xi)]ds_\eta,\ \xi \in \Theta. \end{aligned}$$

Taking into account the inequality (9), we can make the following assessment:

$$\begin{aligned} |W(\xi) - W_\lambda(\xi)| &\leq \left|\int_\Omega[\partial_n W(\eta)\Psi_\lambda(\eta, \Lambda; \xi) - W(\eta)\partial_n \Psi_\lambda(\eta, \Lambda; \xi)]ds_\eta\right| \leq \\ &\leq K\int_\Omega[|\Psi_\lambda(\eta, \Lambda; \xi)| + |\partial_n \Psi_\lambda(\eta, \Lambda; \xi)|]ds_\eta,\ \xi \in \Theta. \end{aligned} \tag{13}$$

roof, we will evaluate the integrals $\int_\Omega|\Psi_\lambda(\eta, \Lambda; \xi)|ds_\eta$, $\int_\Omega\left|\partial_{\eta_1}\Psi_\lambda(\eta, \Lambda; \xi)\right|ds_\eta$ and $\int_\Omega|\partial_n \Psi_\lambda(\eta, \Lambda; \xi)|ds_\eta$ on the part Ω, i.e., on $\eta_2 = 0$.

In order to achieve this, we will choose the imaginary component of identity (7), resulting in the following equation

$$\begin{aligned} \Psi_\lambda(\eta, \Lambda; \xi) = \frac{e^{\sigma(\eta_2 - \xi_2)}}{2\pi}\Bigg[\int_0^\infty \frac{\cos\lambda\sqrt{u^2+\alpha^2}}{u^2+r^2} u\Im_0(\Lambda u)du - \\ - \int_0^\infty \frac{\eta_2 \sin\lambda\sqrt{u^2+\alpha^2}}{u^2+r^2}\frac{u\Im_0(\Lambda u)}{\sqrt{u^2+\alpha^2}}du,\ \eta \neq \xi,\ \xi_2 > 0. \end{aligned} \tag{14}$$

According to (14), along with the inequality,

$$I_0(\Lambda u) \leq \sqrt{\frac{2}{\Lambda\pi u}}, \tag{15}$$

we have the following

$$\int_{\Omega}|\Psi_{\lambda}(\eta,\Lambda;\xi)|ds_{\eta} \leq C(\Lambda,\xi)\lambda e^{-\lambda\xi_2},\ \lambda > 1,\ \xi \in \Theta. \tag{16}$$

At this point, we will evaluate the second integral based on the following considerations

$$\partial_n\Psi_{\lambda}(\eta,\Lambda;\xi) = \partial_{\eta_1}\Psi_{\lambda}(\eta,\Lambda;\xi)\cos\alpha_1 + \partial_{\eta_2}\Psi_{\lambda}(\eta,\Lambda;\xi)\cos\alpha_2). \tag{17}$$

Here $\cos\alpha_1$, $\cos\beta_1$ are the coordinates of the unit external normal n at the point η of the boundary $\partial\Theta$.

Starting from the equality in (14), inequality (15) and applying partial derivatives concerning $\eta_{j,}\ j = \overline{1,2}$, we derive the following approximations:

$$\int_{\Omega}\left|\partial_{\eta_1}\Psi_{\lambda}(\eta,\Lambda;\xi)\right|ds_{\eta} \leq C(\Lambda,\xi)\lambda e^{-\lambda\xi_2},\ \lambda > 1,\ \xi \in \Theta. \tag{18}$$

$$\int_{\Omega}\left|\partial_{\eta_2}\Psi_{\lambda}(\eta,\Lambda;\xi)\right|ds_{\eta} \leq C(\Lambda,\xi)\lambda e^{-\lambda\xi_2},\ \lambda > 1,\ \xi \in \Theta. \tag{19}$$

Ultimately, by merging the approximations from equations (16) to (19) and weaving in (13), we arrive at the approximation (11).

To prove inequality (12), we differentiate equations (8) and (10). In this case, differentiation is taken according to $\xi_j, j = \overline{1,2}$.

$$\begin{gathered}
\partial_{\xi_j}W(\xi) = \int_{\partial\Theta}\left[\partial_n W(\eta)\partial_{\xi_j}\Psi_{\lambda}(\eta,\Lambda;\xi) - W(\eta)\partial_{\xi_j}(\partial_n\Psi_{\lambda}(\eta,\Lambda;\xi))\right]ds_{\eta} = \\
= \int_{\Sigma}\left[\partial_n W(\eta)\partial_{\xi_j}\Psi_{\lambda}(\eta,\Lambda;\xi) - W(\eta)\partial_{\xi_j}(\partial_n\Psi_{\lambda}(\eta,\Lambda;\xi))\right]ds_{\eta} + \\
+ \int_{\Omega}\left[\partial_n W(\eta)\partial_{\xi_j}\Psi_{\lambda}(\eta,\Lambda;\xi) - W(\eta)\partial_{\xi_j}(\partial_n\Psi_{\lambda}(\eta,\Lambda;\xi))\right]ds_{\eta}, \\
\partial_{\xi_j}W_{\lambda}(\xi) = \int_{S}\left[\partial_n W(\eta)\partial_{\xi_j}\Psi_{\lambda}(\eta,\Lambda;\xi) - W(\eta)\partial_{\xi_j}(\partial_n\Psi_{\lambda}(\eta,\Lambda;\xi))\right]ds_{\eta}, \\
\xi \in \Theta,\ j = \overline{1,2}.
\end{gathered} \tag{20}$$

By utilizing the above equation and incorporating inequality (9), we can derive an estimation:

$$\begin{gathered}
\left|\partial_{\xi_j}W(\xi) - \partial_{\xi_j}W_{\lambda}(\xi)\right| \leq \\
\leq \int_{\Omega}\left[\left|\partial_n W(\eta)\partial_{\xi_j}\right||\Psi_{\lambda}(\eta,\Lambda;\xi)| - |W(\eta)|\left|\partial_{\xi_j}(\partial_n\Psi_{\lambda}(\eta,\Lambda;\xi))\right|\right]ds_{\eta} \leq \\
\leq K\int_{\Omega}\left[\left|\partial_{\xi_j}\Psi_{\lambda}(\eta,\Lambda;\xi)\right| + \left|\partial_{\xi_j}\big(\partial_n\Psi_{\lambda}(\eta,\Lambda;\xi)\big)\right|\right]ds_{\eta},\ \xi \in \Theta,\ j = \overline{1,2}.
\end{gathered} \tag{21}$$

For the proof, we will evaluate the integrals $\int_{\Omega}\left|\partial_{\xi_j}\Psi_{\lambda}(\eta,\Lambda;\xi)\right|ds_{\eta}$ and $\int_{\Omega}\left|\partial_{\xi_j}(\partial_n\Psi_{\lambda}(\eta,\Lambda;\xi))\right|ds_{\eta}, j = \overline{1,2}$ on the part Ω of the plane $\eta_2 = 0$.

To estimate the first integral, from equality (14), inequality (15), taking partial derivatives with respect to $\xi_{j,}\ j = \overline{1,2}$, we obtain the following estimates:

$$\int_{\Omega}\left|\partial_{\xi_1}\Psi_{\lambda}(\eta,\Lambda;\xi)\right|ds_{\eta} \leq C(\Lambda,\xi)\lambda e^{-\lambda\xi_2},\ \lambda > 1,\ \xi \in \Theta. \tag{23}$$

$$\int_{\Omega}\left|\partial_{\xi_2}\Psi_{\lambda}(\eta,\Lambda;\xi)\right|ds_{\eta}\leq C(\Lambda,\xi)\lambda e^{-\lambda\xi_2},\ \lambda>1,\ \xi\in\Theta. \tag{24}$$

To assess the secondary integrals, we utilize the equivalence

$$\begin{gathered}\partial_{\xi_j}(\partial_n\Psi_{\sigma}(\eta,\Lambda;\xi))=\partial_{\xi_j}(\partial_{\eta_1}\Psi_{\sigma}(\eta,\Lambda;\xi))\cos\alpha_1+\\+\partial_{\xi_j}(\partial_{\eta_2}\Psi_{\sigma}(\eta,\Lambda;\xi))\cos\beta_1,\ \ j=\overline{1,2}.\end{gathered} \tag{25}$$

As a result, upon examining equations (17) and (25), we arrive at the following approximations:

$$\int_{\Omega}\left|\partial_{\xi_1}(\partial_n\Psi_{\sigma}(\eta,\Lambda;\xi))\right|ds_{\eta}\leq C(\Lambda,\xi)\lambda e^{-\lambda\xi_2},\ \lambda>1,\ \xi\in\Theta. \tag{26}$$

$$\int_{\Omega}\left|\partial_{\xi_2}(\partial_n\Psi_{\sigma}(\eta,\Lambda;\xi))\right|ds_{\eta}\leq C(\Lambda,\xi)\lambda e^{-\lambda\xi_2},\ \lambda>1,\ \xi\in\Theta. \tag{27}$$

At this stage, we combine the assessments from (23) - (24) and (26) - (27), taking into account (21), which enables us to establish the estimation given in (12). □.

Corollary 1. For any $\xi\in\Theta$, the following limit relations are true:

$$\lim_{\lambda\to\infty}W_{\lambda}(\xi)=W(\xi),\ \lim_{\lambda\to\infty}\partial_{\xi_j}W_{\lambda}(\xi)=\partial_{\xi_j}W(\xi),\ \xi\in\Theta,\ j=\overline{1,2}.$$

Assume that $\overline{\Theta}_{\varepsilon}$ consists of a set We denote by $\overline{\Theta}_{\varepsilon}$ the set

$$\overline{\Theta}_{\varepsilon}=\{(\xi_1,\xi_2)\in\Theta,\ a>\xi_2\geq\varepsilon,\ a=\max_{\Omega}\psi(\xi_1),\ 0<\varepsilon<a\}.$$

In this case, $\overline{\Theta}_{\varepsilon}\subset\Theta$ is considered a compact set.

Corollary 2. Suppose $\xi\in\overline{\Theta}_{\varepsilon}$, in this case the families of functions $\{W_{\sigma}(\xi)\}$, $\left\{\partial_{\xi_j}W_{\sigma}(\xi)\right\}$ converge uniformly as $\sigma\to\infty$.

$$W_{\lambda}(\xi)\rightrightarrows W(\xi),\ \partial_{\xi_j}W_{\lambda}(\xi)\rightrightarrows\partial_{\xi_j}W(\xi),\ j=\overline{1,2}.$$

In this case, the set $E_{\varepsilon}=\Theta\backslash\overline{\Theta}_{\varepsilon}$ is a boundary layer for this problem, similar to the theory of singular perturbations, in which uniform convergence cannot exist.

3. Sustainability Assessment

Consider the subsequent equation defined on a smooth curve Σ, where

$$\eta_2=\psi(\eta_1),\ \eta_1\in(-\infty,\infty).$$

Let's put

$$a = \max_{\Omega} \psi(\eta_1),\ b = \max_{\Omega} \sqrt{1 + \left(\frac{d\psi}{d\eta_1}\right)^2}.$$

Theorem 2. Suppose $W(\eta) \in C^2(\Theta) \cap C^1(\Theta)$ satisfy boundary condition (9), and and also on Σ the following inequality

$$|W(\eta)| + |\partial_n W(\eta)| \le \mu,\ \eta \in \Sigma. \tag{28}$$

Concurrently, the subsequent statements are accurate

$$|W(\xi)| \le C(\Lambda, \xi)\lambda K^{1-\frac{\xi_2}{a}} \mu^{\frac{\xi_2}{a}},\ \lambda > 1,\ \xi \in \Theta, \tag{29}$$

$$\left|\partial_{\xi_j} W(\xi)\right| \le C(\Lambda, \xi)\lambda K^{1-\frac{\xi_2}{a}} \mu^{\frac{\xi_2}{a}},\ \lambda > 1,\ \xi \in \Theta,\ j = \overline{1,2}. \tag{30}$$

Proof. To begin with, we will assess inequality (29). Utilizing the integral formulation (8), we derive the ensuing results.

$$\begin{aligned} W(\xi) = &\int_{\Sigma} [\partial_n W(\eta) \Psi_\lambda(\eta, \Lambda; \xi) - W(\eta) \partial_n \Psi_\lambda(\eta, \Lambda; \xi)] ds_\eta + \\ &+ \int_{\Omega} [\partial_n W(\eta) \Psi_\lambda(\eta, \Lambda; \xi) - W(\eta) \partial_n \Psi_\lambda(\eta, \Lambda; \xi)] ds_\eta,\ \xi \in \Theta. \end{aligned} \tag{31}$$

We will assess the equation (28)

$$\begin{aligned} |W(\xi)| \le &\left| \int_{\Sigma} [\partial_n W(\eta) \Psi_\lambda(\eta, \Lambda; \xi) - W(\eta) \partial_n \Psi_\lambda(\eta, \Lambda; \xi)] ds_\eta \right| + \\ &+ \left| \int_{\Omega} [\partial_n W(\eta) \Psi_\lambda(\eta, \Lambda; \xi) - W(\eta) \partial_n \Psi_\lambda(\eta, \Lambda; \xi)] ds_\eta \right|,\ \xi \in \Theta. \end{aligned} \tag{32}$$

By utilizing inequality (29), we begin by approximating the integral with respect to Σ, that is,

$$\begin{aligned} &\left| \int_{\Sigma} [\partial_n W(\eta) \Psi_\lambda(\eta, \Lambda; \xi) - W(\eta) \partial_n \Psi_\lambda(\eta, \Lambda; \xi)] ds_\eta \right| \le \\ &\le \int_{\Sigma} [|\partial_n W(\eta)| |\Psi_\lambda(\eta, \Lambda; \xi)| - |W(\eta)| |\partial_n \Psi_\lambda(\eta, \Lambda; \xi)|] ds_\eta \le \\ &\le \mu \int_{\Sigma} [|\Psi_\lambda(\eta, \Lambda; \xi)| + |\partial_n \Psi_\lambda(\eta, \Lambda; \xi)|] ds_\eta,\ \xi \in \Theta. \end{aligned} \tag{33}$$

For the proof, we will evaluate the integrals $\int_{\Sigma} |\Psi_\lambda(\eta, \Lambda; \xi)| ds_\eta$ and $\int_{\Sigma} |\partial_n \Psi_\lambda(\eta, \Lambda; \xi)| ds_\eta$ on a smooth curve, i.e., on Σ.

Drawing from equation (14) and inequality (15), we derive the following results:

$$\int_{\Sigma} |\Psi_\lambda(\eta, \Lambda; \xi)| ds_\eta \le C(\Lambda, \xi) \lambda e^{\lambda(a - \xi_2)},\ \lambda > 1,\ \xi \in \Theta. \tag{34}$$

Utilizing equations (14) and (17), along with the inequality expressed in (15), we can evaluate the second integral to arrive at the appropriate estimation

$$\int_{\Sigma}\left|\partial_{\eta_1}\Psi_{\lambda}(\eta,\Lambda;\xi)\right|ds_{\eta} \le C(\Lambda,\xi)\lambda e^{\lambda(a-\xi_2)},\ \lambda>1,\ \xi\in\Theta. \tag{35}$$

$$\int_{\Sigma}\left|\partial_{\eta_2}\Psi_{\lambda}(\eta,\Lambda;\xi)\right|ds_{\eta} \le C(\Lambda,\xi)\lambda e^{\lambda(a-\xi_2)},\ \lambda>1,\ \xi\in\Theta. \tag{36}$$

Based on (34) through (36), taking into account (33), we derive

$$\begin{gathered}\left|\int_{\Sigma}[\partial_n W(\eta)\Psi_{\lambda}(\eta,\Lambda;\xi)-W(\eta)\partial_n\Psi_{\lambda}(\eta,\Lambda;\xi)]ds_{\eta}\right| \le \\ \le \mu\int_{\Sigma}[|\Psi_{\lambda}(\eta,\Lambda;\xi)|+|\partial_n\Psi_{\lambda}(\eta,\Lambda;\xi)|]ds_{\eta} \le \\ \le C(\Lambda,\xi)\lambda\mu e^{\lambda(a-\xi_2)},\ \lambda>1,\ \xi\in\Theta.\end{gathered} \tag{37}$$

The following is known

$$\begin{gathered}|W(\xi)-W_{\lambda}(\xi)| \le \left|\int_{\Omega}[\partial_n W(\eta)\Psi_{\lambda}(\eta,\Lambda;\xi)-W(\eta)\partial_n\Psi_{\lambda}(\eta,\Lambda;\xi)]ds_{\eta}\right| \le \\ \le \int_{\Omega}[|\Psi_{\lambda}(\eta,\Lambda;\xi)|+|\partial_n\Psi_{\lambda}(\eta,\Lambda;\xi)|]ds_{\eta} \le \\ \le C(\Lambda,\xi)\lambda K e^{-\sigma\xi_2},\ \lambda>1,\ \xi\in\Theta.\end{gathered} \tag{38}$$

Finally, using the obtained estimates (37) - (38), and also taking into account (32), we obtain the following

$$|W(\xi)| \le \frac{C(\Lambda,\xi)\lambda}{2}\left(\mu e^{\lambda a}+K\right)e^{-\lambda\xi_2},\ \lambda>1,\ \xi\in\Theta, \tag{39}$$

Choosing λ from the equality

$$\lambda=\frac{1}{a}\ln\frac{K}{\mu}, \tag{40}$$

completely we get the estimate (29).

Now we turn to inequality (30). It is required to take partial derivatives of the integral representation (8). Here the integration is carried out according to $\xi_j, j=\overline{1,2}$.

$$\begin{gathered}\partial_{\xi_j}W(\xi)=\int_{\partial\Theta}\left[\partial_n W(\eta)\partial_{\xi_j}\Psi_{\lambda}(\eta,\Lambda;\xi)-W(\eta)\partial_{\xi_j}(\partial_n\Psi_{\lambda}(\eta,\Lambda;\xi))\right]ds_{\eta}= \\ =\int_{\Sigma}\left[\partial_n W(\eta)\partial_{\xi_j}\Psi_{\lambda}(\eta,\Lambda;\xi)-W(\eta)\partial_{\xi_j}(\partial_n\Psi_{\lambda}(\eta,\Lambda;\xi))\right]ds_{\eta}+ \\ +\int_{\Omega}\left[\partial_n W(\eta)\partial_{\xi_j}\Psi_{\lambda}(\eta,\Lambda;\xi)-W(\eta)\partial_{\xi_j}(\partial_n\Psi_{\lambda}(\eta,\Lambda;\xi))\right]ds_{\eta}= \\ =\partial_{\xi_j}W_{\lambda}(\xi)+\int_{\Omega}\left[\partial_n W(\eta)\partial_{\xi_j}\Psi_{\lambda}(\eta,\Lambda;\xi)-W(\eta)\partial_{\xi_j}(\partial_n\Psi_{\lambda}(\eta,\Lambda;\xi))\right]ds_{\eta}, \\ \xi\in\Theta,\ j=\overline{1,2}.\end{gathered} \tag{41}$$

Here

$$\partial_{\xi_j} W_\lambda(\xi) = \int_\Sigma \left[\partial_n W(\eta)\partial_{\xi_j}\Psi_\lambda(\eta,\Lambda;\xi) - W(\eta)\partial_{\xi_j}(\partial_n\Psi_\lambda(\eta,\Lambda;\xi))\right] ds_\eta, \quad \xi\in\Theta,\ j=\overline{1,2}. \tag{42}$$

Based on identity (41), we obtain the following estimate:

$$\begin{aligned}\left|\partial_{\xi_j} W(\xi)\right| &\le \left|\int_{\partial\Theta} \left[\partial_n W(\eta)\partial_{\xi_j}\Psi_\lambda(\eta,\Lambda;\xi) - W(\eta)\partial_{\xi_j}(\partial_n\Psi_\lambda(\eta,\Lambda;\xi))\right] ds_\eta\right| \le \\ &\le \left|\int_\Sigma \left[\partial_n W(\eta)\partial_{\xi_j}\Psi_\lambda(\eta,\Lambda;\xi) - W(\eta)\partial_{\xi_j}(\partial_n\Psi_\lambda(\eta,\Lambda;\xi))\right] ds_\eta\right| + \\ &+ \left|\int_\Omega \left[\partial_n W(\eta)\partial_{\xi_j}\Psi_\lambda(\eta,\Lambda;\xi) - W(\eta)\partial_{\xi_j}(\partial_n\Psi_\lambda(\eta,\Lambda;\xi))\right] ds_\eta\right| \le \\ &\le \left|\partial_{\xi_j} W_\lambda(\xi)\right| + \left|\int_\Omega \left[\partial_n W(\eta)\partial_{\xi_j}\Psi_\lambda(\eta,\Lambda;\xi) - W(\eta)\partial_{\xi_j}(\partial_n\Psi_\lambda(\eta,\Lambda;\xi))\right] ds_\eta\right|, \\ &\xi\in\Theta,\ j=\overline{1,2}.\end{aligned} \tag{43}$$

Taking into account condition (28), we estimate $\left|\partial_{\xi_j} W(\xi)\right|$.

$$\begin{aligned}\left|\partial_{\xi_j} W(\xi)\right| &\le \int_\Sigma \left[|\partial_n W(\eta)|\left|\partial_{\xi_j}\Psi_\lambda(\eta,\Lambda;\xi)\right| - |W(\eta)|\left|\partial_{\xi_j}(\partial_n\Psi_\lambda(\eta,\Lambda;\xi))\right|\right] ds_\eta \le \\ &\le \mu\int_\Sigma \left[\left|\partial_{\xi_j}\Psi_\lambda(\eta,\Lambda;\xi)\right| + \left|\partial_{\xi_j}\big(\partial_n\Psi_\lambda(\eta,\Lambda;\xi)\big)\right|\right] ds_\eta,\ \xi\in\Theta,\ j=\overline{1,2}.\end{aligned} \tag{44}$$

To do this, we estimate the integrals $\int_\Sigma \left|\partial_{\xi_j}\Psi_\lambda(\eta,\Lambda;\xi)\right| ds_\eta$ and $\int_\Sigma \left|\partial_{\xi_j}(\partial_n\Psi_\lambda(\eta,\Lambda;\xi))\right| ds_\eta$, $j=\overline{1,2}$ on a smooth curve Σ.

Based on equality (14) and inequality (17), we have

$$\int_\Sigma \left|\partial_{\xi_1}\Psi_\sigma(\eta,\Lambda;\xi)\right| ds_\eta \le C(\Lambda,\xi)\lambda e^{\lambda(a-\xi_2)},\ \lambda>1,\ \xi\in\Theta. \tag{45}$$

$$\int_\Sigma \left|\partial_{\xi_2}\Psi_\sigma(\eta,\Lambda;\xi)\right| ds_\eta \le C(\Lambda,\xi)\lambda e^{\lambda(a-\xi_2)},\ \lambda>1,\ \xi\in\Theta. \tag{46}$$

To estimate the second integral, based on equality (25), we have

$$\int_\Sigma \left|\partial_{\xi_1}(\partial_n\Psi_\lambda(\eta,\Lambda;\xi))\right| ds_\eta \le C(\Lambda,\xi)\lambda e^{\lambda(a-\xi_2)},\ \lambda>1,\ \xi\in\Theta. \tag{47}$$

$$\int_\Sigma \left|\partial_{\xi_2}(\partial_n\Psi_\lambda(\eta,\Lambda;\xi))\right| ds_\eta \le C(\Lambda,\xi)\lambda e^{\lambda(a-\xi_2)},\ \lambda>1,\ \xi\in\Theta. \tag{48}$$

From (45) – (48), as well as (44), we have

$$\left|\int_\Sigma \left[\partial_n W(\eta)\partial_{\xi_j}\Psi_\lambda(\eta,\Lambda;\xi) - W(\eta)\partial_{\xi_j}(\partial_n\Psi_\lambda(\eta,\Lambda;\xi))\right] ds_\eta\right| \leq \\ \leq C(\Lambda,\xi)\lambda\mu e^{\lambda(a-\xi_2)},\ \lambda > 1,\ \xi \in \Theta,\ j = \overline{1,2}. \tag{49}$$

It is known that we obtained the following estimate in the proof of the previous theorem

$$\left|\int_\Omega \left[\partial_n W(\eta)\partial_{\xi_j}\Psi_\lambda(\eta,\Lambda;\xi) - W(\eta)\partial_{\xi_j}(\partial_n\Psi_\lambda(\eta,\Lambda;\xi))\right] ds_\eta\right| \leq \\ \leq C(\Lambda,\xi)\lambda K e^{-\lambda\xi_2},\ \xi \in \Theta. \tag{50}$$

From (49) - (50), as well as (43), we will have

$$\left|\partial_{\xi_j} W(\xi)\right| \leq \frac{C(\Lambda,\xi)\lambda}{2}\left(\mu e^{\lambda a} + K\right)e^{-\lambda\xi_2},\ \lambda > 1,\ \xi \in \Theta,\ j = \overline{1,2}. \tag{51}$$

In estimate (30), choosing the parameter λ from equality (40), we finally prove completeness (30). □

Suppose $W(\eta) \in C^2(\Theta) \cap C^1(\Theta)$ and instead of functions $f(\eta), g(\eta)$ on a smooth curve Σ be given by their approximations $f_\mu(\eta), g_\mu(\eta)$, respectively, with an error of $0 < \mu < 1$,

$$\max_\Sigma\left|f(\eta) - f_\mu(\eta)\right| \leq \mu,\ \max_\Sigma\left|g(\eta) - g_\mu(\eta)\right| \leq \mu, \tag{52}$$

We put

$$W_{\lambda(\mu)}(\xi) = \int_\Sigma \left[g_\mu(\eta)\Psi_\lambda(\eta,\Lambda;\xi) - f_\mu(\eta)\partial_n\Psi_\mu(\eta,\Lambda;\xi)\right] ds_\eta,\ \xi \in \Theta. \tag{53}$$

Theorem 3. Let $W(\eta) \in C^2(\Theta) \cap C^1(\Theta)$ on the part of the plane $\Omega: \eta_2 = 0$ satisfying condition (9), then the following estimations hold:

$$|W(\xi) - W_\lambda(\xi)| \leq C(\Lambda,\xi)\lambda K^{1-\frac{\xi_2}{a}}\mu^{\frac{\xi_2}{a}},\ \lambda > 1,\ \xi \in \Theta, \tag{54}$$

$$\left|\partial_{\xi_j} W(\xi) - \partial_{\xi_j} W_\lambda(\xi)\right| \leq C(\Lambda,\xi)\lambda K^{1-\frac{\xi_2}{a}}\mu^{\frac{\xi_2}{a}},\ \lambda > 1,\ \xi \in \Theta,\ j = \overline{1,2}. \tag{55}$$

Proof. Taking into account the integral representations (8) and (53), we have

$$W(\xi) - W_{\lambda(\mu)}(\xi) = \int_\Sigma [g(\eta)\Psi_\lambda(\eta,\Lambda;\xi) - f(\eta)\partial_n\Psi_\lambda(\eta,\Lambda;\xi)] ds_\eta + \\ + \int_\Omega [g(\eta)\Psi_\lambda(\eta,\Lambda;\xi) - f(\eta)\partial_n\Psi_\lambda(\eta,\Lambda;\xi)] ds_\eta - \\ - \int_\Sigma \left[g_\mu(\eta)\Psi_\lambda(\eta,\Lambda;\xi) - f_\mu(\eta)\partial_n\Psi_\lambda(\eta,\Lambda;\xi)\right] ds_\eta = \\ = -\int_\Sigma \partial_n\Psi_\lambda(\eta,\Lambda;\xi)\{f(\eta) - f_\mu(\eta)\} ds_\eta + \int_\Sigma \Psi_\sigma(\eta,\Lambda;\xi)\{g(\eta) - g_\mu(\eta)\} ds_\eta + \\ + \int_\Omega [g(\eta)\Psi_\lambda(\eta,\Lambda;\xi) - f(\eta)\partial_n\Psi_\lambda(\eta,\Lambda;\xi)] ds_\eta,\ \xi \in \Theta.$$

and

$$\partial_{\xi_j}W(\xi)-\partial_{\xi_j}W_{\lambda(\mu)}(\xi)=\int_\Sigma\left[g(\eta)\partial_{\xi_j}\Psi_\lambda(\eta,\Lambda;\xi)-f(\eta)\partial_{\xi_j}(\partial_n\Psi_\lambda(\eta,\Lambda;\xi))\right]ds_\eta+$$
$$+\int_\Omega\left[g(\eta)\partial_{\xi_j}\Psi_\lambda(\eta,\Lambda;\xi)-f(\eta)\partial_{\xi_j}(\partial_n\Psi_\lambda(\eta,\Lambda;\xi))\right]ds_\eta-$$
$$-\int_\Sigma\left[g_\mu(\eta)\partial_{\xi_j}\Psi_\sigma(\eta,\Lambda;\xi)-f_\mu(\eta)\partial_{\xi_j}(\partial_n\Psi_\sigma(\eta,\Lambda;\xi))\right]ds_\eta=$$
$$=-\int_\Sigma\partial_{\xi_j}(\partial_n\Psi_\sigma(\eta,\Lambda;\xi))\{f(\eta)-f_\mu(\eta)\}ds_\eta+\int_\Sigma\partial_{\xi_j}\Psi_\lambda(\eta,\Lambda;\xi)\{g(\eta)-g_\delta(\eta)\}ds_\eta+$$
$$+\int_\Omega\left[g(\eta)\partial_{\xi_j}\Psi_\lambda(\eta,\Lambda;\xi)-f(\eta)\partial_{\xi_j}(\partial_n\Psi_\lambda(\eta,\Lambda;\xi))\right]ds_\eta,\ \ \xi\in\Theta,\ \ j=\overline{1,2}.$$

Further, from the boundary condition (9) and conditions (52), we will, respectively, evaluate the following:

$$\left|W(\xi)-W_{\lambda(\mu)}(\xi)\right|\le\int_\Sigma|\partial_n\Psi_\lambda(\eta,\Lambda;\xi)|\left|\{f(\eta)-f_\mu(\eta)\}\right|ds_\eta+$$
$$+\int_\Sigma|\Psi_\sigma(\eta,\Lambda;\xi)|\left|\{g(\eta)-g_\mu(\eta)\}\right|ds_\eta+$$
$$+\int_\Omega[|g(\eta)||\Psi_\lambda(\eta,\Lambda;\xi)|-|f(\eta)||\partial_n\Psi_\lambda(\eta,\Lambda;\xi)|]ds_\eta\le$$
$$\le\mu\int_\Sigma|\partial_n\Psi_\lambda(\eta,\Lambda;\xi)|ds_\eta+\mu\int_\Sigma|\Psi_\sigma(\eta,\Lambda;\xi)|ds_\eta+$$
$$+K\int_\Omega[|\Psi_\lambda(\eta,\Lambda;\xi)|-|\partial_n\Psi_\lambda(\eta,\Lambda;\xi)|]ds_\eta,\ \xi\in\Theta.$$

and

$$\left|\partial_{\xi_j}W(\xi)-\partial_{\xi_j}W_{\lambda(\mu)}(\xi)\right|\le\int_\Sigma\left|\partial_{\xi_j}(\partial_n\Psi_\sigma(\eta,\Lambda;\xi))\right|\left|\{f(\eta)-f_\mu(\eta)\}\right|ds_\eta+$$
$$+\int_\Sigma\left|\partial_{\xi_j}\Psi_\lambda(\eta,\Lambda;\xi)\right|\left|\{g(\eta)-g_\delta(\eta)\}\right|ds_\eta+$$
$$+\int_\Omega\left[|g(\eta)|\left|\partial_{\xi_j}\Psi_\lambda(\eta,\Lambda;\xi)\right|-|f(\eta)|\left|\partial_{\xi_j}(\partial_n\Psi_\lambda(\eta,\Lambda;\xi))\right|\right]ds_\eta\le$$
$$\le\mu\int_\Sigma\left|\partial_{\xi_j}(\partial_n\Psi_\sigma(\eta,\Lambda;\xi))\right|ds_\eta+\mu\int_\Sigma\left|\partial_{\xi_j}\Psi_\lambda(\eta,\Lambda;\xi)\right|ds_\eta+$$
$$+K\int_\Omega\left[\left|\partial_{\xi_j}\Psi_\lambda(\eta,\Lambda;\xi)\right|-\left|\partial_{\xi_j}(\partial_n\Psi_\lambda(\eta,\Lambda;\xi))\right|\right]ds_\eta,\ \ \xi\in\Theta,\ \ j=\overline{1,2}.$$

From the results of the theorems obtained above, we obtain, respectively, the following estimates:

$$\left|W(\xi)-W_{\lambda(\mu)}(\xi)\right|\le\frac{C(\Lambda,\xi)\sigma}{2}\left(\mu e^{\lambda a}+K\right)e^{-\sigma\xi_2},\ \lambda>1,\ \xi\in\Theta,\tag{56}$$

$$\left|\partial_{\xi_j}W(\xi)-\partial_{\xi_j}W_{\lambda(\mu)}(\xi)\right|\le\frac{C(\Lambda,\xi)\sigma}{2}\left(\mu e^{\lambda a}+K\right)e^{-\sigma\xi_2},\quad \lambda>1,\ \xi\in\Theta,\ j=\overline{1,2}.\tag{57}$$

In the last obtained estimates (56) and (57), choosing the parameter λ, from (40) respectively, we obtain the proof of Theorem 3. □

Corollary 3. We claim that for any $\xi\in\Theta$, the following limit equalities hold:

$$\lim_{\mu\to0}W_{\lambda(\mu)}(\xi)=W(\xi),\ \lim_{\mu\to0}\partial_{\xi_j}W_{\lambda(\mu)}(\xi)=\partial_{\xi_j}W(\xi),\ \xi\in\Theta,\ j=\overline{1,2}.$$

Corollary 4. It turns out that if $\xi \in \overline{\Theta}_\varepsilon$, then the families of functions $\{W_{\lambda(\mu)}(\xi)\}$ and $\left\{\partial_{\xi_j} W_{\lambda(\mu)}(\xi)\right\}$ converge uniformly at $\mu \to 0$

$$W_{\lambda(\mu)}(\xi) \rightrightarrows W(\xi),\ \partial_{\xi_j} W_{\lambda(\mu)}(\xi) \rightrightarrows \partial_{\xi_j} W(\xi),\ \xi \in \Theta,\ j = \overline{1,2}.$$

4. Conclusion

In summary, this research utilizes the Carleman function to reconstruct an unknown function using Cauchy data provided on a designated segment of the boundary within the area. By developing the Carleman function and implementing Green's formula, a clear regularized solution can be achieved. The findings indicate that the efficient construction of the Carleman function is tantamount to creating a regularized solution for the Cauchy problem. It is postulated that a smooth solution exists within a closed set with well-defined Cauchy data. Under this assumption, explicit expressions for extending the solution and its derivative are derived, alongside a regularization formula applicable when continuous approximations of the initial Cauchy data are given with a specific error in the uniform metric. Additionally, stability estimates concerning the solution of the Cauchy problem in a traditional sense are also established. Moreover, the stability estimates derived from the analysis not only affirm the robustness of the solution but also establish a framework for quantitative assessments of the Cauchy problem's sensitivity to perturbations in the input data. Such insights are crucial for practitioners who rely on accurate model predictions in dynamic systems. The approach ensures that even when faced with errors in the Cauchy data, a reliable recovery of the original function can be achieved, reinforcing the reliability of this method in critical analyses.

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